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Lorentz Spaces and Lie Groups

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This paper is motivated by the behavior of the heat diffusion kernel $p_t(x)$ on a general unimodular Lie group. Indeed, contrary to what happens in $\mathbb{R}^n$, the $P_t(x)$ on a general Lie group is behaving like $t^{-\delta(t)/2}$ for two possibly distinct integers $\delta(t)$, one for t tending to 0 and another for t tending to ∞, namely d and D. This forces us to consider a natural generalization of Lorentz spaces with different indices at "zero" and at "infinity."

0. INTRODUCTION

0.1. Let G be a C^∞ connected manifold and $H=\{X_1, ..., X_k\}$ be C^∞ vector fields on G. We shall say that the system H satisfies the Hörmander condition or H is a Hörmander system if together with their successive brackets $[X_{a_1}, [X_{a_2}, [\cdots X_{a_s}]\cdots]]$ they span at every point of G the tangent space of G (see [1]).

Now let $l(t)\in G$, $0\leqslant t\leqslant 1$ be an absolutely continuous path on G such that

$$\dot{l}(t)=dl\left(\frac{\partial}{\partial t}\right)=\sum_{j=1}^{k} a_j(t)\,X_j \qquad (\text{a.e. } t\in[0,1]).$$

Setting $|l|=\int_0^1 \{\sum_{j=1}^k |a_j(t)|^2\}^{1/2}\,dt$ for two points $x, y\in G$, we define

$$d(x, y)=d_H(x, y)=\inf\{|l|: l(0)=x, l(1)=y\}, \tag{0.1}$$

where the inf is taken over all the paths that satisfy the above condition. It is well known that $d(\cdot,\cdot)$ is a distance function on G which induces the canonical topology on G (see [2, 3]).

0.2. Let G be a connected Lie group and g its Lie algebra generated by a Hormander system of left invariant vector fields. We define $B_t=\{x\in G: d(x, e)<t\}$, the ball of radius t contered at the point $e\in G$.

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Also we define the volume of B_t as $V(t)=\mu(B_t)$, where μ is a left invariant Haar measure on G.

0.3. It can be proved that there exists a number $d \in N$ and a constant $c>0$ s.t.

$$C^{-1}t^d \leqslant V(t) \leqslant ct^d, \qquad 0<t<1 \qquad \text{(see [4])}.$$

This d is called the local dimension or the dimension at "zero" and it depends on the choice of the vector fields. On the other hand according to the well-known theorem of Y. Guivarc'h either there is $D>0$ and $c>0$ s.t.

$$C^{-1}t^D \leqslant V(t) \leqslant ct^D, \qquad t \geqslant 1,$$

or there are $c_1, c_2>0$ s.t. $V(t)>c_1 e^{c_2 t}$, $t \geqslant 1$. Note that in the case $t \geqslant 1$, c depends only on G and not on the choice of vector fields (see [5]).

Remark. In the first case we say that G is of polynomial growth and it has dimension at infinity D. In the second case we say that G is of exponential growth and its dimension at infinity is $D=+\infty$. In the special case of simply connected nilpotent groups we have $d \leqslant D$.

0.4. Suppose $p_t(x, y)$ is the fundamental solution of the equation $(\partial/\partial t+\Delta)\, u=0$, where $\Delta=\sum_{j=1}^k X_j^2$ and X_j is as in subsection 0.1.

The potential operator $\Delta^{-a/2}$ is defined by a convolution operator as follows:

$$\Delta^{-a/2}f=\int K_a(x, y)\, f(y)\, dy, \tag{0.4.1}$$

where

$$K_a(x, y)=\int_0^\infty t^{-a/2-1}p_t(x, t)\, dt. \tag{0.4.2}$$

$\Delta^{-a/2}$ can also be defined as the negative fractional powers of Δ by spectral theorem (see [6]).

0.5. Recently N. Varopoulos has proved the following theorem (see [7]).

THEOREM. *Let G a connected Lie group of polynomial growth and H a left invariant Hörmander system on G. Then $\Delta^{-a/2}$ is a bounded operator from L^p tp $L^{pn/(n-ap)}$ iff $d \leqslant D$, $n \in [d, D]$, $l<p<+\infty$, and $0<ap<n$.*

The above is noting more than a generalization of the classical Hardy–Littlewood theorem on $\mathbb{R}^n$ (see [6]). This classical theorem has another generalization due to R. Hunt (see [8]) which is connected with convolution operators on Lorentz spaces on $\mathbb{R}^n$. In this paper we propose to generalize Varopoulos' theorem in the context of the Lorentz space on the groups that are adapted to the different behavior of p_t. These Lorentz spaces need two different sets of indices, one pair for the zero and one for infinity.

0.6. Let f be a scalar valued function defined on a measure space (G, μ). We define the distribution function of f, $\lambda_f(y)=\mu\{x\in G\colon |f(x)|>y\}$. With each function f we associate the rearrangement function of f, $f^*(t)=\inf\{y>0\colon \lambda_f(y)<t, t>0\}$.

DEFINITION. The Lorentz space $L^{q_1 q_2}_{p_1 p_2}$ is the collection of all f such that $\|f\|^{q_1 q_2}_{p_1 p_2}<\infty$, where

$$\|f\|^{q_1 q_2}_{p_1 p_2}=\begin{cases}\left(\dfrac{q_1}{p_1}\displaystyle\int_0^1 \left(t^{1/p_1} f^*(t)^{q_1}\frac{dt}{t}\right)\right)^{1/q_1}+\left(\dfrac{q_2}{p_2}\displaystyle\int_1^\infty \left(t^{1/p_2} f^*(t)\right)^{q_2}\frac{dt}{t}\right)^{1/q_2}, \\ \qquad p_1, p_2, q_1, q_2<\infty \\ \sup\limits_{t<1} t^{1/p_1} f^*(t)+\sup\limits_{t>1} t^{1/p_2} f^*(t), \qquad q_1=q_2=\infty \\ \left(\dfrac{q_1}{p_1}\displaystyle\int_0^1 (t^{1/p_1} f^*(t))^{q_1}\frac{dt}{t}\right)^{1/q_1}+\sup t^{1/p_2} f^*(t), \qquad q_2=\infty \\ \sup\limits_{t<1} t^{1/p_1} f^*(t)+\left(\dfrac{q_2}{p}\displaystyle\int_1^\infty (t^{1/p_2} f^*(t))^{q_2}\frac{dt}{t}\right)^{1/q_2}, \qquad q_1=\infty.\end{cases}$$

It is easy to see that $L^{qq}_{pp}=L_{(p,q)}$, the known Lorentz spaces, and $L^{pp}_{pp}=L_p$, the Lebesque spaces. Moreover, if $p_1<p_2$ Homstedt's formula for the K-functional of the pair $(L_{(p_1,q_1)}, L_{(p_2,q_2)})$ at $t=l$ is exactly the quasi-norm of the space $L^{q_1 q_2}_{p_1 p_2}$ (see [9, 10]).

Here we must mention a similar device which has been used for Besov spaces, for the same reason, in [11].

0.7. We are now in a position to state the main theorem of this paper.

THEOREM 0. *Let G be a unimodular nonactomic connected Lie group and f a scalar-valued function on G. Then the potential operator $\Delta^{-a/2}$ is a bounded mapping from $L^{q_1' q_2'}_{p_1' p_2'}$ to $L^{q_1 q_2}_{p_1 p_2}$, where*

$$\frac{1}{p_1}=\frac{1}{p_1'}+\frac{1}{p_1''}-1, \qquad \frac{1}{p_2}=\frac{1}{p_2'}+\frac{1}{p_2''}-1, \qquad \frac{1}{q_1}=\frac{1}{q_1'}+\frac{1}{q_1''}\leqslant 1$$

$$\frac{1}{q_2}=\frac{1}{q_2'}+\frac{1}{q_2''}\leqslant 1, \qquad p_1''\leqslant\frac{1}{d-a}, \qquad p_2''\geqslant\frac{1}{D-a}, \qquad 1<q, q_2, q_1', \quad q_2'<\infty$$

$$1<p_i, \qquad p_i', p_i''<\infty, \qquad i=1, 2, \ldots$$

If G has exponential volume growth then $p_2''\geqslant 1/(N-a)$.

0.8. In Section 1 of this paper we give an inequality that holds for Lorentz spaces as well as the topological properties of these spaces. In Section 2 we give two interpolation theorems analogous to Marcinkiewicz (weak-type) and Riesz–Thorin (strong-type) interpolation theorems. In the Section 3 we prove a convolution theorem which has as its corollary Theorem 0. Finally, I express my gratitude to Professors N. Varopoulos and G. Alexopoulos for their invaluable help in the preparation of this paper.

1. Inequalities and Topological Properties

1.1. The functional $f \to \|f\|_{pr}^{qs}$ is not always a norm, even when $p, q, r, s \geqslant 1$. We can turn L_{pr}^{qs} into a normed space, as in Lorentz spaces $L_{p,q}$, if we replace f^* with the maximal operator of f^*, say f^{**}, in the definition (0.6), for $1 < p, q, r, s \leqslant \infty$. Recall that $f^{**}(t) = (1/t) \int_0^t f^*(s)\, ds$, $t > 0$, and it is known that $f^* \leqslant f^{**}$.

So, setting $\|f^{**}\|_{pr}^{qs} = |f|_{pr}^{qs}$ we can say that the normed space L_{pr}^{qs} consists of all functions f, defined as in (0.6) for which the quantity $|f|_{pr}^{qs}$ is finite, since the following result holds.

Proposition 1.1. *If $1 < p$, $r \leqslant \infty$ and $1 \leqslant q$, $s \leqslant \infty$ then*

$$\|f\|_{pr}^{qs} \leqslant |f|_{pr}^{qs} \leqslant c\, \|f\|_{pr}^{qs}.$$

In particular $(L_{pr}^{qs}, |f|_{pr}^{qs})$ is a normed space.

Proof. The first inequality is an immediate consequence of the definitions of the quantities $\|f\|_{pr}^{qs}$ and $|f|_{pr}^{qs}$ and the fact that $f^* < f^{**}$. The second follows from Hardy's inequality. Since $f \to f^{**}$ is subadditive, the triangle inequality for $|f|_{pr}^{qs}$ follows immediately from Minkowski's inequality. For more details see [8, 9].

1.2. Let $\mathscr{P}$ be the family of all operators $P = P_s$ of the form $Pf = X_s \cdot f$, where S is any measurable set of measure 1, X_s denotes the characteristic function of S, and f is as in (0.6). Let $\mathscr{A}$ be the set of all sequences $A = \{A_n\}_{n \in \mathbb{N}}$ of pairwise disjoint measurable sets each of measure 1 and for each such A let Q_A be the operator of "shifted" conditional expectation $Q_A f = \sum_n \int_{An} f\, d\mu \cdot X_{An+1}$. Let $\mathscr{Q}$ be the family of all such operators. Since $Q_A f$ is a step function, its decreasing rearrangement is a step function of the form $(Q_A f)^*(t) = \sum_{j \geqslant 1} b_j X(t)_{[j-1, j]}$, $0 < t \leqslant \infty$, because $\mu(A_j) = 1$.

Note that $\{b_j\}_{j \geqslant 1}$ is the decreasing rearrangement of the vector $\{a_i\}_{i \geqslant 1}$, where $a_i = \int_{A_i} f\, d\mu$, supposing, without loss of generality, that f is a non-negative function. In the sequel we will prove a proposition which is the key proposition for the study of the L_{pr}^{qs} spaces. This enables us to deduce some properties of L_{pr}^{qs} spaces immediately from those of the L_{pq} spaces. Moreover this observation enables the interpolation theorems of Section 2 required in this paper to be deduced almost immediately from those of Hunt for Lorentz spaces. We assume that all measure spaces throughout the paper are nonatomic.

PROPOSITION 1.2. *If* $1 < p, r, s, r \leqslant \infty$, *and* $f \in L_{pr}^{qs}$ *then and only then* $Pf \in L_{pq}$ *and* $Qf \in L_{rs}$ *for all* $p \in \mathscr{P}$ *and* $Q \in \mathscr{Q}$. *In particular the norm in* L_{pr}^{qs} *is equivalent to* $\sup_{P \in \mathscr{P}} \|Pf\|_{pq} + \sup_{Q \in \mathscr{Q}} \|Qf\|_{rs}$ *and the supremum is attained to within some fixed multiplicative constant for some suitable choices of P and Q.*

Proof. First we will show that $(Q_A f)^{**}(t) \leqslant f^{**}(t)$, $t \geqslant 0$. To obtain this result it suffices to show that

$$\int_0^t (Q_A f)^*(s)\,ds \leqslant \int_0^t f^*(s)\,ds, \qquad \text{for all} \quad t \geqslant 0. \tag{1.2.1}$$

The above formula is also an immediate consequence of two very standard facts in interpolation theory which can be found in [10], namely Theorem 5.2.1 (p. 109) in the case $p = 1$ and Eq. (8), p. 41. (Obviously Q_A maps L^p into L^p with norm 1 for $p = 1$ and $p = \infty$ so we have $M_0 = M_1 = 1$ in that equation.) But

$$\begin{aligned}\int_0^t (Q_A f)^*(s)\,ds &= \int_0^t \sum b_j X(s)_{[j-1,j]}\,ds = \int_0^1 b_1\,ds + \int_1^2 b_2\,ds + \cdots \\ &\quad + \int_{[t]}^t b_{[t]+1}\,ds \\ &\leqslant \int_{B_1} f\,d\mu + \int_{B_2} f\,d\mu + \cdots + \int_{B_{[t]}} f\,d\mu \\ &\quad + \int_0^{t-[t]} (f X_{B[t]+1})^*(s)\,ds \qquad (*)\end{aligned}$$

since

$$\begin{aligned}\int_{[t]}^t b_{[t]+1}\,ds &= (t-[t]) \int_{B[t]+1} f\,d\mu \leqslant (t-[t])(f X_{B[t]+1})^{**}(\mu(B_{[+]+1})) \\ &\leqslant \int_0^{t-[t]} (f X_{B[t]+1})^*(s)\,ds\end{aligned}$$

because f^{**} is a decreasing function, where $[t]$ symbolizes the integral part of t.

It is also known that if (M, μ) is a finite nonatomic measure space, f a positive function on M, and λ any number satisfying $0 \leqslant \lambda \leqslant \mu(M)$ then there is a measurable set E_λ, with $\mu(E_\lambda) = \lambda$ such that $\int_{E_\lambda} f\, d\mu = \int_0^\lambda f^*(s)\, ds$.

Using the above, relation $(*)$, and the fact that $\mu(B_{[t]+1}) = 1$ we obtain

$$\int_0^t (Q_A f)^*(s)\, ds \leqslant \int_{B_1 UB_2 UB_{[t]} UE_{t-[t]}} f\, du \leqslant \int_0^t f^*(s)\, ds$$

for $t > 0$. Note that B_j is the domain of the integral b_j. Moreover

$$\left(\frac{s}{r}\int_0^1 (t^{1/r}(Qf)^*(t))^s \frac{dt}{t}\right)^{1/s} = b_1 = \left(\frac{q}{p}\int_0^1 (t^{1/p}(Qf)^*(t))^q \frac{dt}{t}\right)^{1/q}$$
$$\leqslant \left(\frac{q}{p}\int_0^1 (t^{1/p}(Qf)^{**}(t))^q \frac{dt}{t}\right)^{1/q}$$
$$\leqslant \left(\frac{q}{p}\int_0^1 (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}$$
$$\leqslant \left(\frac{q}{p}\int_0^1 \left(t^{1/p}\frac{1}{t}\int_0^t f^*(s)\, ds\right)^q \frac{dt}{t}\right).$$

Using Hardy's inequality we have

$$\left(\frac{s}{r}\int_0^1 (t^{1/p}(Qf)^*(t))^s \frac{dt}{t}\right)^{1/s} \leqslant c\left(\frac{q}{p}\int_0^1 (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}$$

or

$$\|Qf\|_{rs} \leqslant c\, \|f\|_{pr}^{qs}. \tag{1.2.2}$$

Now suppose $t < 1$; then $f^*(t) = \inf\{\lambda : \mu\{|f| > \lambda\} \leqslant t \leqslant 1\} \geqslant \inf\{\lambda : \mu\{|fx_s| > \lambda\} \leqslant t < 1\} = (fX_s)^*(t)$, where $\mu(S) = 1$. So

$$\left(\frac{q}{p}\int_0^1 (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} \geqslant \left(\frac{q}{p}\int_0^1 (t^{1/p}(P_s f)^*(t)^q \frac{dt}{t}\right)^{1/q} \tag{1.2.3}$$

for every set S of measure 1.

Furthermore if one chooses $\bar{A}_n = \{x \in G : f^*(n-1) \geqslant |f(x)| \geqslant f^*(n)\}$, $n \geqslant 1$, and we put $\bar{A} = \{\bar{A}_n\}_{n \in \mathbb{N}}$ and $\bar{a}_i = \int_{\bar{A}_i} f\, d\mu$ we have

$$(Q_{\bar{A}} f)^*(t) = \sum_{i \geqslant 1} \bar{a}_i X(t)_{[i-1,\, i]}, \qquad t > 0$$

and

$$f^*(i) \leqslant \bar{a}_i \leqslant f^*(i-1), \qquad i \in \mathbb{N}.$$

So

$$\left(\frac{s}{r}\int_{n+1}^{n+2} (t^{1/r} f^*(t)^s \frac{dt}{t}\right)^{1/s} \leqslant c \left(\frac{s}{r}\int_{n}^{n+1} (t^{1/r}(Q_{\bar{A}} f)^* \,(t))^* \frac{dt}{t}\right)^{1/s}.$$

By a change of variables we deduce

$$\left(\frac{s}{r}\int_{1}^{\infty} (t^{1/r} f^*(t))^s \frac{dt}{t}\right)^{1/s} \leqslant c \, \|Q_{\bar{A}} f\|_{rs}. \tag{1.2.4}$$

Also, if $\bar{A}_1 = \{x \in G \colon |f(x)| > f^*(1)\}$ it is easy to see that

$$\left(\frac{q}{p}\int_{0}^{1} (t^{1/p} f^*(t)^q \frac{dt}{t}\right)^{1/q} = \sup_{p \in \mathscr{P}} \|Pf\|_{pq}. \tag{1.2.5}$$

Combining relations (1.2.2), (1.2.3), (1.2.4), and (1.2.5) we obtain the desired result.

1.3. The next result shows that, for any fixed p and r. the Lorentz space L_{pr}^{qs} increases as the upper exponents q and r increase; i.e., there are inclusion relations among L_{pr}^{qs} spaces, with q and s varying, like those for the Lebesgue spaces or Lorentz spaces $L_{p,\,q}$.

PROPOSITION 1.3. *Suppose* $0 < p,\ r \leqslant \infty,\ 0 < q_1 \leqslant q_2 < \infty,$ *and* $0 < s_1 \leqslant s_2 \leqslant \infty$. *Then* $\|f\|_{pr}^{q_2 s_2} \leqslant c\, \|f\|_{pr}^{q_1 s_1}$, *where* c *in a constant depending on* p, r, q_i *and* s_i, i, 1, 2. *In particular, there is the embedding* $L_{p,\,r}^{q_1 s_1} \to L_{p,\,r}^{q_2 s_2}$.

Proof. The proof follows directly from the fact that Lorentz spaces $L_{p,\,q}$ increase as the secondary exponent q increases and by Proposition 1.2.

Inclusion relations among L_{pr}^{qs}, with p and r, are like those for the Lebesgue spaces L^p and Lorentz spaces in that they depend on the structure of the underlying measure space. The upper exponents are not involved. Thus, if $0 < p_1 \leqslant p_2 \leqslant \infty$, $0 < r_1 \leqslant r_2 \leqslant \infty$, and $0 < q,\ r \leqslant \infty$, then $L_{p_2 r_2}^{qr} \to L_{p_1 r_1}^{qr}$ on finite measure spaces. The above is an immediate consequence of the properties of the L_{pq} spaces and the fact that on a finite measure space $L_{pr}^{qs} = L_{pq}$.

1.4. In the sequel of this section we will give some topological properties of L_{pr}^{qs} spaces. It is easy to see that L_{pr}^{qs} spaces equipped with the functional $e(f, g) = |f - g|_{pr}^{qs}$ are metric spaces since $[(f+g)^{**}\,(t)]^r \leqslant [f^{**}(t)]^r + [g^{**}(t)]^r$, $0 \leqslant r \leqslant 1$. So it can be proved that the L_{pr}^{qs} spaces, equipped with the metric e, are complete spaces. Moreover if $r=1$, e is a norm. This norm is applicable to the L^{qs}_{pr} spaces when $1<p,\ r\leqslant\infty$ and $1\leqslant q,\ s\leqslant\infty$. Hence we can conclude the following result.

PROPOSITION 1.4. *If $1<p,\ r\leqslant\infty$ and $1\leqslant p,\ s\leqslant\infty$ the L^{qs}_{pr} spaces are Banach spaces for any measure space (G,μ).*

Before proceeding to the interpolation theorems of this paper, let us determine the conjugate spaces of the L^{qs}_{pr} spaces, which will play a particularly important role in the convolution theorems cited in the final section of this paper.

1.5. We proceed two propositions.

PROPOSITION 1.5.1. *The space $L^{\infty\,\infty}_{p_1\,p_2}$ is the conjugate space of $L^{1,\,1}_{p_1'p_2'}$, where $1/p_1+1/p_1'=1/p_2+1/p_2'=1$ and f and g on (G,μ).*

Proof. (a)

$$\|fg\|_1=\int_G |fg|\,d\mu\leqslant\int_0^\infty f^*(t)\,g^*(t)\,dt=\int_0^1 f^*(t)\,g^*(t)\,dt$$

$$+\int_0^\infty f^*(t)\,g^*(t)\,dt$$

$$=\int_0^1 t^{1/p_1'}g^*(t)\,t^{1/p_1}f^*(t)\,\frac{dt}{t}+\int_1^\infty t^{1/p_2'}g^*(t)\,t^{1/p_2}f^*(t)\,\frac{dt}{t}$$

$$\leqslant\sup_{t<1}t^{1/p_1'}g^*(t)\int_0^1 t^{1/p_1}f^*(t)\,\frac{dt}{t}+\sup_{t\geqslant 1}t^{1/p_2'}g^*(t)\int_0^\infty t^{1/p_2}f^*(t)\,\frac{dt}{t}$$

$$+\sup_{t<1}t^{1/p_1'}g^*(t)\int_0^\infty t^{1/p_2}f^*(t)\,\frac{dt}{t}+\sup_{t<1}t^{1/p_2'}g^*(t)\int_1^1 t^{1/p_1}f^*(t)\,\frac{dt}{t}$$

$$=\|g\|^{\infty\ \infty}_{p_1'\ p_2'}\,\|f\|^{1\ \ 1}_{p_1\,p_2}.$$

(b) Define $m(s)=l(x_s)$, where X_s is the characteristic function of the set s, $m(s)$ is a measure, and $|m(s)|\leqslant B_i\,\|X_s\|^{1\ \ 1}_{p_1\,p_2}=B_i(\mu(s))^{1/p_1'}$, where $i=1$ if $m(s)<1$ and $i=2$ if $m(s)\geqslant 1$. l is a continuous linear functional in L^{qs}_{pr} and $|l(f)|\leqslant B\,\|f\|^{qs}_{pr}$ for every $f\in L^{qs}_{pr}$. Hence m is absolutely continuous with respect to μ. Then the Randon–Nikodym theorem (see [14]) gives a function $g(x)$ such that $m(s)=l(X_s)=\int_s g(x)\,d\mu$ and hence $\int_G g(x)\,f(x)\,d\mu\leqslant B\,\|f\|^{1\ \ 1}_{p_1\,p_2}\ \forall f\in L^{qs}_{pr}$.

Setting $f(x)=[\exp(-i\arg g(x))]\,X_s$ we obtain $\int_s |g(x)|\,d\mu \leqslant B[\mu(s)]^{1/p_i}$. Therefore

$$\frac{1}{\mu(s)}\int_S |g(x)|\,d\mu \leqslant B_i[m(s)]^{-1/p_i'} \leqslant B_i t^{-1/p_i'}.$$

It follows that $g^{**}(t)\leqslant B_1 t^{-1/p_1'}$ if $t<i$. So $g\in L^{\infty\;\infty}_{p_1'\;p_2'}$ and $\|f\|^{1\;1}_{p_1\,p_2}\leqslant B$.

Note that the inverse is not expected as in Lebesgue spaces or in Lorentz spaces L_{pq}.

PROPOSITION 1.5.2. *The conjugate space of* $L^{q_1q_2}_{p_1p_2}$ *is* $L^{q_1'q_2'}_{p_1'p_2'}$, *where*

$$\frac{1}{p_1}+\frac{1}{p_1'}=\frac{1}{p_2}+\frac{1}{p_2'}=\frac{1}{q_1}+\frac{1}{q_1'}=\frac{1}{q_2}+\frac{1}{q_2'}=1$$

and

$$1<p_1, p_2, q_1, q_2<\infty.$$

Proof. Let $f\in(L^{q_1q_2}_{p_1p_1})^*$. Since $\|f\|^{q_1q_2}_{p_1p_2}\leqslant c\|f\|^{1\;1}_{p_1\,p_2}$, $L\in(L^{1\;1}_{p_1\,p_1})^*$; hence there exists $g\in L^{\infty\;\infty}_{p_1'\;p_2'}$ such that $l(f)=\int_G f(x)\,g(x)\,d\mu$, $\forall f\in L^{1\;1}_{p_1\,p_2}(*)$.

Using the fact that $|l(f)|\leqslant B\,\|f\|^{q_1q_2}_{p_1p_2}$ it can be shown that $g\in L^{q_1'q_2'}_{p_1'p_2'}$ and $(*)$ holds for every $f\in L^{q_1q_2}_{p_1p_2}$. Conversely, for any $g\in L^{q_1'q_2'}_{p_1'p_2'}$, $(*)$ defines a continuous linear functional on $L^{q_1q_2}_{p_1p_2}$.

2. INTERPOLATION THEOREMS

2.1. We next wish to prove two interpolation theorems which are generalizations of the interpolation theorems of Marcinkiewicz and Riesz–Thorin. Both theorems are useful for the convolution results which will be shown in Section 3. Before proceeding to the theorems we will give some definitions concerning operators which are involved in the theorems that follow.

DEFINITION 2.1.1. An operator T mapping functions on a measure space into functions on another space is called quasi-linear if $T(f+g)$ is defined whenever Tf and Tg are defined and if $|T(f+g)|\leqslant k(T(f)+T(g))$ a.e., where k is a constant independent of f and g.

DEFINITION 2.1.2. An operator T, as in Definition 2.1.1, is called sublinear if whenever Tf and Tg are defined and c is a constant then

$T(f+g)+T(cf)$ are defined with $|T(f+g)| \leqslant |Tf| + |Tg|$ and $|T(cf)| = |c|\ |T(f)|$. Note that it follows $|\,|Tf| - |Tg|\,| \leqslant |T(f-g)|$.

2.2. Let $f \in L_{pr}^{qs}$ so $Pf \in L_{pq}$ and $Qf \in L_{rs}$, where Pf and Qf are as in Proposition 1.2. By definition of Pf we have that $Pf \in L_{pr}^{qs}$ and by relations (1.2.1) and (1.2.2) we have that $Qf \in L_{pr}^{qs}$. Now let u be a function defined on the domain of f and taking values in $[0, 1]$. It is easy to see that if $Qf \in L_{rs}$ then $uQf \in L_{rs}$ and hence $uQf \in L_{pr}^{qs}$. Suppose $\bar{A} = \{A_n\}_{n \in \mathbb{N}}$ as in the proof of Proposition 1.2 and $u(x) = 0$ if $x \in A_1$ and $u(x) = f(x)/\bar{a}$ if $x \in \bar{A}_{n+1}$. Since $f \cdot X_{A_{n+1}} \leqslant f^*(n) \leqslant \int_{A_n} f\, d\mu$ we see that $u(x) \leqslant 1$, provided that $f \in L^1(An)$ for all n. From these we have the following result.

PROPOSITION 2.2.1. *Any non-negative function f in $L^1 + L^\infty$ can be expressed in the form $f = Pf + uQf$ for some suitable choices of P and Q, as defined in* (1.2), *where u is some function taking values in* $[0, 1]$.

2.3. Now we are able to proceed to the first interpolation theorem which is of Marcinkiewicz type.

WEAK-TYPE THEOREM 2.3.1. *If T is a quasi-linear operator and for all $f \in L_{p_i r_i}^{q_i s_i}$ we have*

$$\|Tf\|_{p_i' r_i'}^{q_i' s_i'} \leqslant B_i \|f\|_{p_i r_i}^{q_i s_i}, \qquad i = 0, 1,$$

then $\|Tf\|_{p'r'}^{q's'} \leqslant B_\theta \|f\|_{pr}^{qs}$ for all $f \in L_{pr}^{qs}$, where B_θ is a constant depending on the exponents, $q \leqslant q'$, $s \leqslant s'$, $p_0 < p_1$, $q_0 < q_1$, $q \leqslant s'$, $s \leqslant q'$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1},$$

$$\frac{1}{p'} = \frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}, \qquad \frac{1}{r'} = \frac{1-\theta}{r_0'} + \frac{\theta}{r_1'},$$

and $0 < \theta < 1$.

Proof. Using the quasi-linearity of T and Proposition 1.2 we have $\|Tf\|_{p'r'}^{q's'} \leqslant k(\|TPf\|_{r'r'}^{q's'} + \|Tu\,Qf\|_{p'r'}^{q's'}) \leqslant \bar{k}(\|P_1 TPf\|_{p'q'} + \|u_1 Q_1 TPf\|_{r's'} + \|P_2 Tu\,Qf\|_{p'q'} + \|u_2 Q_2 Tu\,Qf\|_{r's'})$, where P_i and Q_i are operators constructed as P and Q, respectively and they correspond to the set families $\bar{A}^i$, $i = 1, 2$.

Note that $\bar{A}^1$ is the sequence of sets which corresponds to the function TPf, i.e., $A_1^1 = \{x \colon |TPf(x)| > (TPf)^*\,(1)\}$ and $A_n^1 = \{x \colon (TPf)^*\,(n-1) \geqslant |TPf(x)| > (TPf)^*\,(n)\}$. Similarly $\bar{A}^2$ is the sequence of sets which is constructed from the function $Tu\,Qf$. The functions $u_1(x)$ and $u_2(x)$ are constructed analogously to $u(x)$; i.e. $u_1(x) = 0$ if $x \in A_1^i$ and $u_1(x) = TPf(x)/\bar{a}_2^1$

if $x \in A^1_{n+1}$, where $\bar{a}^1_n = \int_{\bar{A}^1_n} TPf\, d\mu$ and $u_2(x) = Tu(x)\, Qf(x)/\bar{a}^2_n$ if $x \in A^2_{n+1}$, where $\bar{a}^2_n = \int_{\bar{A}^2_n} Tu(x)\, Qf(x)\, d\mu$.

Observe that u_1 and u_2 are functions taking values in $[0, 1]$. Using the observation at the beginning of 2.2 and the fact that $u(x) \leqslant 1$ and $u_i(x) \leqslant 1$ it is easy to see that $P_1\, TP$ and $P_2\, TuQ$ are quasi-linear operators from $L_{p_i q_i}$ to $L_{p'_i q'_i}$ and $Q_1\, TP$ and $Q_2\, TuQ$ are quasi-linear operators from $L_{r_i s_i}$ to $L_{r'_i s'_i}$ for $i = 0, 1$.

So by Hunt's weak-type interpolation theorem (see [8]) we obtain

$$\|P_1\, TPf\|_{p'q'} \leqslant H_\theta\, \|f\|_{pq}$$

$$\|P_2\, Tu\, Qf\|_{p'q'} \leqslant J_\theta\, \|f\|_{rs}$$

$$\|Q_1\, TPf\|_{r's'} \leqslant L_\theta\, \|f\|_{pq}$$

$$\|Q_2\, Tu\, Qf\|_{r's'} \leqslant K_\theta\, \|f\|_{rs}.$$

Hence $\|Tf\|^{q's'}_{p'r'} \leqslant C_\theta\{\|f\|_{pq} + \|f\|_{rs}\}$. Using again Proposition 1.2 we have the desired result. Here we would like to inform the reader that these two theorems can also be perhaps more easily understood and more easily proved in a general context of interpolation theory, as for example in [9, 10].

STRONG-TYPE-THEOREM 2.3.2. *Suppose T is a sublinear operator such that*

$$\|Tf\|^{q_i s_i}_{p_i r_i} \leqslant B_i\, \|f\|^{q'_i s'_i}_{p'_i r'_i}, \qquad i = 0, 1.$$

Then $\|Tf\|^{q\,s}_{p\,r} \leqslant B_\theta\, \|f\|^{q'\,s'}_{p'\,r'}$, *where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1},$$

$$\frac{1}{p'} = \frac{1-\theta}{p'_0} + \frac{\theta}{p'_1}, \qquad \frac{1}{r'} = \frac{1-\theta}{r'_0} + \frac{\theta}{r'_1},$$

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad \frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1},$$

$$\frac{1}{q'} = \frac{1-\theta}{q'_0} + \frac{\theta}{q'_1}, \qquad \frac{1}{s'} = \frac{1-\theta}{s'_0} + \frac{\theta}{s_i}, \qquad 0 < \theta < 1.$$

For the proof of the above theorem we follow the proof of the weak-type interpolation theorem using the strong-type interpolation theorem of Hunt (see [8]) in the place of the weak-type one.

3. A Convolution Theorem on $L^{q_1 q_2}_{p_1 p_2}$

3.1. Suppose G is locally compact unimodular connected Lie group and μ is a Haar measure on G. The convolution of two functions f and g is defined by $f{*}g(x) = \int_G f(y)\, g(xy^{-1})\, d\mu(y)$ provided that the integral exists. So we have the following

Theorem 3.1. *$\|f{*}g\|^{q_1 q_2}_{p_1 p_2} \leqslant B\, \|f\|^{q'_1 q'_2}_{p'_1 p'_2}\, \|g\|^{q''_1 q''_2}_{p''_1 p''_2}$, where B is constant and*

$$\frac{1}{p_1} = \frac{1}{p'_1} + \frac{1}{p''_1} - 1, \qquad \frac{1}{q_1} = \frac{1}{q'_1} + \frac{1}{q''_1} \leqslant 1,$$

$$\frac{1}{p_2} = \frac{1}{p'_2} + \frac{1}{p''_2} - 1, \qquad \frac{1}{q_2} = \frac{1}{q'_2} + \frac{1}{q''_2} \leqslant 1$$

for $1 < p_1, p_2, p'_1, p'_2, p''_1, p''_2 < \infty$.

Proof. Using Hölder's inequality, it can be shown that $L^{q_1 q_2}_{p_1 p_2}$ have the multplicative property, i.e.,

$$\|fg\|^{q_1 q_2}_{p_1 p_2} \leqslant c\, \|f\|^{q'_1 q'_2}_{p'_1 p'_2}\, \|g\|^{q''_1 q''_2}_{p''_1 p''_2},$$

where

$$\frac{1}{p_1} = \frac{1}{p'_1} + \frac{1}{p''_1}, \qquad \frac{1}{p_2} = \frac{1}{p'_2} + \frac{1}{p''_2},$$

$$\frac{1}{q_1} = \frac{1}{q'_1} + \frac{1}{q''_1}, \qquad \frac{1}{q_2} = \frac{1}{q'_2} + \frac{1}{q''_2}.$$

Moreover, it is true that

$$\|f{*}g\|^{\infty\ \infty}_{p_1\ p_2} \leqslant c\, \|f\|_1\, \|g\|^{\infty\ \infty}_{p^*_1\ p^*_2}, \qquad 1 < p_1, \quad p_2 < \infty, \tag{3.1.1}$$

and

$$\|f{*}g\|_\infty \leqslant c\, \|f\|^{1\ \ 1}_{p'_1\, p'_2}\, \|g\|^{\infty\ \infty}_{p_1\ p_2}, \qquad 1 < p_1, \quad p_2 < \infty, \tag{3.1.2}$$

where

$$\frac{1}{p_1} + \frac{1}{p^*_1} = 1 \qquad \text{and} \qquad \frac{1}{p_2} + \frac{1}{p^*_2} = 1.$$

Applying the weak-type interpolation theorem to (3.1.1) and (3.1.2). We have that

$$\|f{*}g\|^{q'_1 q'_2}_{p_1 p_2} \leqslant c\, \|f\|^{q'_1 q'_2}_{p'_1 p'_2}\, \|g\|^{\infty\ \infty}_{p''_1\ p''_2}, \tag{3.1.3}$$

where

$$\frac{1}{p_1}=\frac{1}{p_1'}+\frac{1}{p_1''}-1<1, \qquad \frac{1}{p_2}=\frac{1}{p_2'}+\frac{1}{p_2''}-1<1, \quad 1\leqslant q_1', \quad q_2'\leqslant\infty$$

and

$$1<p_1', p_2', p_1'', p_2''<\infty.$$

By Proposition 1.5.1 we have that

$$\|f^*g\|_{p_1\,p_2}^{1\;1}=\sup_{\|h\|_{\bar p_1\bar p_2}^{\infty\,\infty}<1} B\int (f^*g)(x)\,h(x)\,d\mu(x),$$

where

$$1/p_1+1/\bar p_1=1 \qquad \text{and} \qquad 1/p_2+1/\bar p_2=1, \qquad \|h\|\leqslant 1;$$

i.e., $h^*(t)\leqslant t^{1/p_1^{-1}}$ if $t\leqslant 1$ and $h^*(t)\leqslant t^{1/p_2^{-1}}$ if $t\geqslant 1$.

Let $I(h)\int (f^*g)(x)\,h(x)\,d\mu(x)$; then

$$|I(h)|\leqslant\int\left(\int |f(y)|\,|g(xy^{-1})|\,d\mu(y)\right)|h(x)|\,d\mu(x)$$

$$=\int |f(y)|\left(\int |g(xy^{-1})|\,d\mu(y)\right)|h(x)|\,d\mu(x).$$

Hence, $|I(h)|\leqslant\|fk\|_1$, where $k(y)=\int |g(xy^{-1})|\,|h(x)|\,d\mu(x)$.

From the multiplication theorem it follows that $|I(h)|\leqslant B\,\|f\|_{p_1'p_2'}^{q_1'q_2'}\,\|k\|_{\lambda\;\;k}^{q_1^*\;q_2^*}$, where

$$\frac{1}{p_1'}+\frac{1}{\lambda}=\frac{1}{p_2'}+\frac{1}{k}=1, \qquad \frac{1}{q_1'}+\frac{1}{q_1^*}=1, \qquad \frac{1}{q_2'}+\frac{1}{q_2^*}=1.$$

But $k=|\bar g|_*\,|h|$, where $\bar g(x)=g(x^{-1})$. Hence by (3.1.3) we have $\|k\|_{\lambda\;k}^{q_1^*q_2^*}\leqslant B\,\|g\|_{p_1''p_2''}^{q_1^*q_2^*}\,\|h\|_{\bar p_1\;\bar p_2}^{\infty\;\infty}\leqslant B\,\|\bar g\|_{p_1''p_2''}^{q_1^*q_2^*}$, where

$$\frac{1}{\bar p_1}=2-\frac{1}{p_1'}-\frac{1}{p_1''} \qquad \text{and} \qquad \frac{1}{\bar p_2}=2-\frac{1}{p_2'}-\frac{1}{p_2''}.$$

Since $(G, d\mu)$ is unimodular, we have $\bar g^{-*}(t)=g^*(t)$ and this gives

$$\|f^*g\|_{p_1\,p_2}^{1\;1}\leqslant B\,\|f\|_{q_1'p_2'}^{q_1'q_2'}\,\|g\|_{p_1''p_2''}^{q_1''q_2''}, \tag{3.1.4}$$

where

$$\frac{1}{p_1}=\frac{1}{p'_1}+\frac{1}{p''_1}-1<1, \qquad \frac{1}{p_2}=\frac{1}{p'_2}+\frac{1}{p''_2}-1<1,$$

$$\frac{1}{q'_1}+\frac{1}{q^*_1}=1, \qquad \frac{1}{q'_2}+\frac{1}{q^*_2}=1$$

for $1<p'_1, p''_1, p'_2, p''_2<\infty$. Finally, using the strong-type interpolation theorem in (3.1.3) and (3.1.4) we obtain Theorem 3.1.

The referee was kind enough to show me an alternative proof of the above theorem. The proof is more direct but uses ideas that are more sophisticated and are inspired from R. O'Neil's work [15].

3.2. *Proof of Theorem* 0. We shall need the following estimates for the heat kernel $p_t(x)$ which can be found in [12]. Let us use the notation $|x|=d(e,x)$. Then if G has polynomial volume growth then there are constants $c_1, c_2, c_3, c_4>0$ such that

$$c_1\, V(t)^{-1/2}\, e^{-|x|^2/c_2 t} \leqslant p_t(x) \leqslant c_3\, V(t)^{-1/2}\, e^{-|x|^2/c_3 t}. \tag{3.2.1}$$

If G has exponential volume growth, then (3.2.1) is still valid for $0\leqslant t\leqslant 1$ and for $t>1$ we have that for $\forall N>0$ there are again constants $c_3, c_4>0$ such that

$$p_t(x)\leqslant c_3\, t^{-N/2} e^{-|x|^2/c_4 t} \tag{3.2.2}$$

Let us recall that there is $c>0$ such that

$$c^{-1} t^D \leqslant V(t) \leqslant ct^D, \qquad t\geqslant 1, \tag{3.2.3}$$

when G has polynomial volume growth, that $V(t)\geqslant c_1 e^{c_2 t}$ for $c_1, c_2>0$ and $t\geqslant 1$ when G has exponential volume growth and that for small t we have

$$c^{-1} t \leqslant V(t) \leqslant ct^d, \qquad 0<t<1, \tag{3.2.4}$$

regardless of the volume growth of the group G. Putting (0.4.2), (3.2.1), (3.2.3), and (3.2.4) we get that, when G has polynomial volume growth,

$$K_a(x) \leqslant \int_0^1 c_1\, t^{-a/2-1-d/2} e^{-|x|^2/c_2 t}\, dt + \int_1^\infty c_3\, t^{-a/2-1-D/2} e^{-|x|^2/c_4 t}\, dt$$

and from this by an easy calculation that

$$K_a(x) \leqslant \begin{cases} \dfrac{x}{|x|^{d-a}}, \ |x| \leqslant 1 \\ \dfrac{c}{|x|^{D-a}}, |x| \geqslant 1, \end{cases}$$

which in term gives that

$$K_a^*(x) \leqslant \begin{cases} cs^{a-d}, & 0 < s \leqslant 1 \\ cs^{a-D}, & c \leqslant 1, \end{cases}$$

where $K_a^*(x)$ is the rearrangment function of $K_a(x)$. From the above we have that $K_a(x) \in L^{q_1'' q_2''}_{p_1'' p_2''}$ if $p_1'' \in 1/(d-a)$ and $p_2'' \in 1/(D-a)$, which with convolution theorem proves Theorem 0 when G has polynomial volume growth.

Similarly, when G has exponential volume growth, (0.4.2), (3.2.1), (3.2.2), and (3.2.4) give that for $\forall N > 0$ $K_a(x) \in L^{q_1'' q_2''}_{p_1'' p_2''}$. If $p_1'' \leqslant 1/(d-a)$ and $p_2'' \geqslant 1/(N-a)$ which with convolution theorem (3.1) proves Theorem 0 for this case.

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